ON CANONICAL VARIABLES IN INTEGRABLE MODELS OF MAGNETS

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1. INTRODUCTION

The most general formulation of phenomenological models of magnets (or spin systems) which includes all the known completely integrable ones, has the following form:

$$\mathbf{S}_t = \mathbf{F}_0(x, y, \mathbf{S}, \mathbf{S}_x, \mathbf{S}_y, \mathbf{S}_{xx}, \mathbf{S}_{yy}, \mathbf{S}_{xy}, \mathbf{J}, u, u_x, u_y, u_{xy}, \alpha^2),$$
$$u_{xx} + \alpha^2 u_{yy} = R_0(\mathbf{S}, \mathbf{S}_x, \mathbf{S}_y), \tag{1.1}$$

where $\mathbf{S} = \mathbf{S}(x, y, t)$ is the magnetization vector, $\mathbf{F}_0(,)$ is a vector-function, $u = u(x, y)$ is an auxiliary field, $R_0(,)$ is a scalar function, $\mathbf{J}$ is a set of constants characterizing the magnet, and $\alpha^2 = \pm 1$.

The function $\mathbf{F}_0$ usually takes the form:

$$\mathbf{F}_0 = \mathbf{S} \wedge \frac{\delta F_{eff}}{\delta \mathbf{S}} + \mathbf{F}_1, \tag{1.2}$$

where F_{eff} is the functional of the crystal's free energy (throughout the paper the symbol δ/δ stands for the variational derivative). The first term in the right hand side was suggested by Landau and Lifshits [1] to describe the exchange interactions.

The representation (1.1)-(1.2) is often inconvenient for solving problems. One would like to deal with more tractable forms of the equations (1.1), which, in turn, requires introduction of new dependent variables. Apparently, such a variable, the stereographic projection, has been used for the first time in paper [2] to describe the instanton solutions in the two-dimensional $O(3)$ σ-model (the $2D$ stationary Heisenberg ferromagnet). Later it was exploited in various situations, see, e. g. [3-5].

In the present paper we show on examples of three models - the deformed Heisenberg, the Landau - Lifshits, and the Ishimori magnets - that it is helpful to introduce the corresponding canonical variables. In particular, they allow to simplify significantly certain calculations, as compared to the usage of the $\mathbf{S}$ variable, and, more substantially, to clarify a set of questions important both from physical and mathematical viewpoints. Another argument in their favor is that in these variables the models fit in a class of models admitting a differential-geometric interpretation intensively studied recently [6].

The model of deformed Heisenberg magnet was suggested in [7] where also an exact solution of it for the case of trivial background was obtained by the inverse scattering method, and the conservation laws were calculated. In doing so, it was shown that

perturbations localized in the space are spreaded, that is, the solutions are instable. The gauge equivalence of this model and the nonlinear Schrödinger equation with an integral nonlinearity was established in [8] and [9]. The matrix Darboux transform method was applied in the paper [10], where exact solutions of the model where calculated on the background of new spiral-logarithmic structures.

The Landau - Lifshits equation is a subject of vast studies. In particular, the Lax representation and conservation laws for it in the completely anisotropic case have first been obtained in [11], soliton solutions were found by the dressing method in [12]. Hamilton aspects of the equation were analyzed in detail in recent paper [13].

The Ishimori magnet was also considered in many papers. In particular, series of exact solutions were obtained in [14] by the inverse scattering and $\bar{\partial}$ - dressing methods, the Darboux transform was applied to it in [15] and [16] (in [16] - on the background of spiral structures). Notice also the important paper [5], where the gauge equivalence of the Ishimori-II and Davy-Stewartson-II models was established.

The structure of this paper is as follows. In section 2 we consider the deformed Heisenberg magnet model, define the canonical variables and analyze stability of the solutions. In section 3 the Landau - Lifshits equation is obtained in terms of the stereographic projection and a stationary version of this equation is studied. Finally, in section 4 we define two pairs of canonical variables for the Ishimori model, re-write the model and the Hamiltonian in these variables, and calculate the Hamiltonian on some of the simplest known solutions. This is preceded by a discussion of the physical interpretation of the model. The Appendix contains a Lax pair for an "extended" system of the deformed Heisenberg magnet model.

2. DEFORMED HEISENBERG MAGNET EQUATION

a). *Canonical variables.*

Let us consider the deformed Heisenberg magnet equation [7][1]:

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{xx} + \frac{1}{x}\mathbf{S} \wedge \mathbf{S}_x. \tag{2.1}$$

Here $x = \sqrt{x_1^2 + x_2^2} > 0$, x_1, x_2 are the Cartesian coordinates on the plane, $\mathbf{S}(x,t) = (S_1, S_2, S_3)$, $|\mathbf{S}| = 1$.

The phase space for this equation is generated by initial data (S_1, S_2, S_3) subject to the constraint $|\mathbf{S}| = 1$. The Poisson brackets of the canonical variables S_i in the model satisfy the standard relations:

$$\{S_i(x), S_j(y)\} = -\varepsilon_{ijk} S_k(x)\delta(x-y), \quad i,j,k = 1,2.3, \tag{2.2}$$

where ϵ_{ijk} is the fully antisymmetric third rank tensor. For any two functionals F, G we then have

$$\{F, G\} = -\int_0^\infty \epsilon_{ijk} S_k \frac{\delta F}{\delta S_i(x)} \frac{\delta G}{\delta S_j(x)}\, dx. \tag{2.3}$$

[1]This equation can be thought of as a cylindrical-symmetric reduction of the (2+1)-dimensional non-integrable Landau - Lifshits equation, $\mathbf{S}_t = \mathbf{S} \wedge \triangle \mathbf{S}$. The relation between the latter and the system of coupled nonlinear Schrödinger equations in the dimension (2+1) has been discussed in detail in [17].

On taking into account (2.2)-(2.3), one can represent the equation (2.1) in the following Hamiltonian form:

$$\mathbf{S}_t = \frac{1}{x}\{H, \mathbf{S}\}, \tag{2.4}$$

where the Hamiltonian H is given by

$$H = \frac{1}{2}\int_0^\infty x\mathbf{S}_x^2\, dx. \tag{2.5}$$

Let us now define a new dependent complex-valued variable,

$$w(x,t) = \frac{S_1 + iS_2}{1 - S_3}, \tag{2.6}$$

which is, at each fixed moment of time t, the stereographical projection of the unit sphere onto the complex plane, $w:\ \mathbb{S}^2 \to \mathbb{C}\cup\{\infty\}$.

In terms of this variable the equation (2.1) can be rewritten as

$$iw_t = w_{xx} - 2\frac{w_x^2\bar{w}}{1+|w|^2} + \frac{1}{x}w_x, \tag{2.7}$$

and the Poisson brackets corresponding to (2.2) take the form[2]

$$\{w(x), w(y)\} = \{\bar{w}(x), \bar{w}(y)\} = 0,\ \ \{w(x), \bar{w}(y)\} = -\frac{i}{2}(1+|w|^2)^2\delta(x-y). \tag{2.8}$$

The bracket (2.3) then becomes

$$\{F, G\} = -\frac{i}{2}\int dx(1+|w(x)|^2)^2\Big[\frac{\delta F}{\delta w(x)}\frac{\delta G}{\delta \bar{w}(x)} - \frac{\delta F}{\delta \bar{w}(x)}\frac{\delta G}{\delta w(x)}\Big], \tag{2.9}$$

and the evolution of the system will be described by the equation

$$iw_t = -\frac{1}{2x}(1+|w|^2)^2\frac{\delta H}{\delta \bar{w}(x)}, \tag{2.10}$$

with the Hamiltonian

$$H = 2\int_0^\infty x\frac{w_x\bar{w}_x}{(1+|w|^2)^2}\, dx. \tag{2.11}$$

It should be noticed that the following "complex extension" of the system (2.1) is of interest of its own [3],

$$ir_t = r_{xx} - \frac{2r_x^2 s}{1+rs} + \frac{1}{x}r_x, \quad is_t = -s_{xx} + \frac{2s_x^2 r}{1+rs} - \frac{1}{x}s_x. \tag{2.12}$$

From (2.8) we obtain the Poisson brackets of variables r и s in the form

[2] In the derivation of (2.8) we use the relations $\{S_\pm(x), S_\pm(y)\} = 0$, $\{S_+(x), S_3(y)\} = -iS_+(x)\delta(x-y)$, $\{S_+(x), S_-(y)\} = 2iS_3(x)\delta(x-y)$, where $S_\pm = S_1 \pm iS_2$, and the Leibnits's rule.

[3] In absence of the nonlinear component the second equation in (2.12) can be interpreted as the free Shrödinger equation with an effective mass. It is evident then that the first equation can be obtained from the second by complex conjugation.

$$\{r(x), s(y)\} = -i(1+rs)^2\delta(x-y), \quad \{\bar{r}(x), \bar{s}(y)\} = i(1+\bar{r}\bar{s})^2\delta(x-y). \tag{2.13}$$

The system (2.12), as well as equation (2.7), is completely integrable (see Appendix) and have a Hamiltonian structure with the Hamiltonian

$$H = \int_0^\infty x \frac{r_x s_x}{(1+rs)^2}\, dx \tag{2.14}$$

and the equations of motion

$$r_t = \frac{1}{x}\{H, r\}, \quad s_t = -\frac{1}{x}\{H, s\}, \tag{2.15}$$

and can be considered a model of the system of two coupled deformed Heisenberg's magnets.

The Poisson brackets (2.13) can be found from the expression for symplectic two-form,

$$\Phi = i\int_0^\infty \Big[\frac{dr \wedge ds}{(1+rs)^2} - \frac{d\bar{r} \wedge d\bar{s}}{(1+\bar{r}\bar{s})^2)}\Big] dx, \quad \Phi = d\varphi, \tag{2.16}$$

where

$$\varphi = -i\int_0^\infty \Big[\frac{ds}{s(1+rs)} - \frac{d\bar{s}}{\bar{s}(1+\bar{r}\bar{s}))}\Big] dx, \tag{2.17}$$

thus (2.16) and (2.17) agree with the corresponding expressions obtained in [18] for the standard Heisenberg magnet.

Notice also that the equation (2.7) is a bi-Hamiltonian system:

$$i\begin{pmatrix} w \\ \bar{w} \end{pmatrix}_t = G_1 \begin{pmatrix} \frac{\delta H_1}{\delta w} \\ \frac{\delta H_1}{\delta \bar{w}} \end{pmatrix} = G_2 \begin{pmatrix} \frac{\delta H_2}{\delta w} \\ \frac{\delta H_2}{\delta \bar{w}} \end{pmatrix}, \tag{2.18}$$

where H_1 coincides with H given by (2.11), the second Hamiltonian H_2 reads as

$$H_2 = -i\int_0^\infty x \frac{w_x \bar{w} - \bar{w}_x w}{(1+|w|^2)|w|^2}\, dx, \tag{2.19}$$

and $G_1 = G_1(w, \bar{w})$, $G_2 = G_2(w, \bar{w})$ are the so-called Hamiltonian operators of the form

$$G_1 = \frac{1}{x(1+|w|^2)^2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{2.20}$$

An expression for the matrix operator G_2 can be obtained from results in paper [19] on the standard Heisenberg magnet but is too cumbersome to be written here. Let us just mention that its matrix entries contain a differential and an integral operator thus rendering it non-local.

The relations (2.18)-(2.20) mean that the recursion operator of the equation (2.7) under the assumption that $det\, G_2 \neq 0$ is represented in the form

$$R = G_1 G_2^{-1}. \tag{2.21}$$

Since (2.7) is a completely integrable system, it admits infinitely many integrals of motion, $\{I_n\}_{n=1}^{\infty}$ [7], in involution, that is, satisfying $\{I_j, I_k\} = 0$. In turn, this allows to obtain hierarchies of the Poisson structures,

$$I_n = RI_{n-1}, \tag{2.22}$$

and the higher equations of the deformed Heisenberg magnet ($j = 0, 1, ...; t_0 = t$),

$$iw_{t_j} = R^j G_2 \frac{\delta H_2}{\delta \bar{w}}. \tag{2.23}$$

b). *Stability of certain solutions of equation (2.7).*

The problem of stability of stationary solutions of the equation (2.7) is of interest since the equation contains the independent variable x explicitly. To analyze it, let $w = w_{st} + \tilde{w}$. On linearizing (2.7), first on the trivial background $w_{st} = 0$, which corresponds, in terms of the magnetization vector, to the vector $\mathbf{S} = (0, 0, 1)$, we obtain:

$$i\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) + \frac{1}{x}\tilde{w}_x(x,t). \tag{2.24}$$

Suppose that $(x > 0)$

$$\tilde{w}(x,0) = \tilde{w}_0(x), \quad \tilde{w}(0,t) = \tilde{w}_1(t). \tag{2.25}$$

Then the equation (2.24) can be solved by the Laplace transformation in the t variable under the additional assumption that $|\tilde{w}(x,t)| < Me^{s_0 t}$ with an $M > 0$ and $s_0 \geq 0$. Solving the arising equation and performing the inverse transformation we find:

$$\bar{w}(x,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{p\,t}[C_0(x,p))J_0(\sqrt{-ip}\,x)]\,dp, \tag{2.26}$$

where $J_0(.)$ is the Bessel function,

$$C_0(x,p) = -i\int_0^x e^{-\int_0^x Q(\xi)\,d\xi}\,[\int_0^x \frac{\tilde{w}_0(y)}{J_0(\sqrt{-ip}\,y)}\,e^{\int_0^y Q(s)\,ds}\,dy\,]\,dx, \tag{2.27}$$

$$Q(x) = -2\sqrt{-ip}\,(\ln J_0(\sqrt{-ip}\,x))_x + \frac{1}{x},$$

$\mathrm{Re}\,a > 0$, the path of integration is any straight line $\mathrm{Re}\,p = a > s_0 > 0$, and the integral in (2.26) is understood in the sense of the principal value. It is not difficult to see that the logarithmic divergencies arising in the exponentials when integrating at the lower limit in (2.28), cancel each other.

It follows from (2.26) that for a fixed x the function $|\tilde{w}(x,t)|$ grows with the t increase, and, as in [7], we obtain that the solution is unstable[4]: an arbitrary localized initial perturbation of the system can grow indefinitely as the time passes.

We now proceed to analyze stability of the stationary state $w_{st} = ie^{i\theta(x)}$ where $\theta(x) = \ln(x) + \theta_0$, $\theta_0 \in \mathbb{R}$ is a constant. This solution is an example of a spiral-logarithmic

[4]Of course, the stability of that linearized "non-autonomous" equation is meant.

structure found in [10]: $\mathbf{S} = (\sin\theta,\ \cos\theta,\ 0)$ [5]. On linearizing the equation (2.26) on this background, we have,

$$i\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) + \frac{1}{x}\tilde{w}_x - i\frac{e^{-i\theta(x)}}{x^2}. \tag{2.29}$$

This equation only differs from (2.24) by the presence of a non-homogeneous term. Hence, its general solution is a sum of (2.26) and a partial solution. It follows that it will be unstable as well.

Notice then, that the equation (2.7) admits a solution periodic in t of the form $w(x,t) = W(x)e^{ikt}$ with k a real constant, provided that the equation[6]

$$W_{xx} - \frac{2W_x^2\bar{W}}{1+|W|^2} + \frac{1}{x}W_x + kW = 0 \tag{2.30}$$

has a solution. This suggests that the study of the linearized stability is insufficient. The analysis of the nonlinear stability requires more subtle methods [see, e. g. [20] and literature cited therein].

3. LANDAU-LIFSHITS MAGNET

a). *Canonical variables.*

The fully anisotropic model of Landau-Lifshits has the form [7]

$$\mathbf{S}_t = \mathbf{S}\wedge\mathbf{S}_{xx} + \mathbf{S}\wedge J\mathbf{S}, \tag{3.1}$$

where $J = diag(J_1, J_2, J_3)$ are diagonal 3×3 matrices, and $J_1,\ J_2,\ J_3$ are parameters of the anisotropy, $J_1 < J_2 < J_3$.

The Hamiltonian for (3.1) can be written in the form,

$$H = \frac{1}{2}\int_{-\infty}^{\infty}(\mathbf{S}_x^2 - \mathbf{S}J\mathbf{S})dx, \tag{3.2}$$

or, using the variable w defined in (2.6), as [8]

$$H = \int_{-\infty}^{\infty}\Big(\frac{2(|w|_x^2 + \alpha(w^2+\bar{w}^2) - \gamma|w|^2)}{(1+|w|^2)^2} - \beta\Big)dx, \tag{3.3}$$

where

[5] Using (2.11), it is easy to check that the Hamiltonian logarithmically diverges on this solution in both limits and, thus, requires a regularization.

[6] Removing the nonlinear term we obtain here the stationary Schrödinger equation with the Coulomb potential and an effective mass.

[7] It is well-known [21], that this model is one of the most general completely integrable models admitting 2×2 -matrix Lax representations.

[8] We assume here that w is a slowly decreasing function. In the case of a decreasing w one should add $J_3 = 4\beta$ to the density of the Hamiltonian.

$$\alpha = \frac{J_2 - J_1}{4}, \;\; \beta = \frac{J_3}{4}, \;\; \gamma = J_3 - \frac{J_1 + J_2}{2}. \tag{3.4}$$

Taking into account (2.9), from this we obtain the following equation of motion for Landau-Lifshits magnet model [9],

$$iw_t = i\{H, w\} = -\frac{1}{2}(1 + |w|^2)^2 \frac{\delta H}{\delta \bar{w}}, \tag{3.5}$$

or [10]

$$iw_t = w_{xx} - 2\frac{\bar{w}(w_x^2 + \alpha) - \alpha w^3 - \gamma w}{1 + |w|^2} - \gamma w. \tag{3.6}$$

Let us consider an implication of this form of the equation. Obvious transformations lead to the following relation which contains the parameter α only,

$$i(|w|^2)_t = (w_x \bar{w} - w \bar{w}_x)_x + 2\frac{w^2 \bar{w}_x^2 - \bar{w}^2 {w_x}^2}{1 + |w|^2} + 2\alpha(w^2 - \bar{w}^2). \tag{3.7}$$

Letting $w = \rho e^{i\varphi}$, где $\rho = \rho(x,t), \; \varphi = \varphi(x,t), \; \rho, \; \varphi \in \mathbb{R}$, we obtain:

$$(\rho^2)_t = 2(\rho^2 \phi_x)_x - \frac{8\rho^3 \rho_x \phi_x}{1 + \rho^2} + 4\alpha \rho^2 \sin 2\phi. \tag{3.8}$$

Defining the variables $R = \rho^2$ и $Q = 2\rho^2 \phi_x$, we now find the following "conservation law":

$$R_t = Q_x - 2Q[\ln(1 + R)]_x + 4\alpha R \sin(\int_{-\infty}^{x} \frac{Q}{R}) \, dx. \tag{3.9}$$

It is especially simple when $\alpha = 0$, which corresponds to the anisotropy of "the easy plan" type.

In a way similar to (2.18), one can produce a bi-Hamiltonian structure for (3.6) with H_1 equal to H defined by (3.3), the Hamiltonian

$$H_2 = \int_{-\infty}^{\infty} \frac{w_x \bar{w} - \bar{w}_x w}{(1 + |w|^2)|w|^2} \, dx, \tag{3.10}$$

and the Hamiltonian operators

$$G_1 = \frac{1}{(1 + |w|^2)^2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{3.11}$$

and G_2 being a matrix integro-differential operator [22]. In terms of the variables w и $\bar{w}$ the recursion operator can be written as follows:

$$R = G_1 G_2^{-1}. \tag{3.12}$$

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[9] Notice that, as well as in the case of the deformed Heisenberg magnet, we are able to obtain the corresponding complex extension (see, [18]); we are not going to dwell on that here.

[10] On taking the complex conjugated equation and neglecting the nonlinear component, one can obtain the nonstationary Shrödinger equation with the potential $V = -\gamma = const$.

It produces an hierarchy of the Poisson structures similar to (2.22) and higher Landau-Lifshits equations similar to (3.5).

b). *The Dispersion relation. Stationary Landau-Lifshits equation.*

Linearizing the equation complex conjugate to (3.6) and choosing $\bar{w} = \bar{w}(x,t)$ in the form $\bar{w} \sim \exp\{i(kx - \omega t)\}$, we have,

$$\omega = k^2 - \gamma, \tag{3.13}$$

which gives a dispersion relation for the Landau-Lifshits equation which is typical for magnets with an exchange interaction [23]. In our case the group and phase velocities are given by $v_g = \partial\omega/\partial k = 2k$, $v_{ph} = \omega/k = k - \gamma/k$, respectively (the latter is infinite for $k = 0$), implying that there is a dispersion in the system. The propagation of a magnetization wave in this model is possible under the condition $k^2 > \gamma = J_3 - (J_1 + J_2)/2 > 0$.

Letting $w = w(x - \mu t) = w(\xi)$ in (3.6) , where $\mu = const$ is the velocity of a stationary profile wave, we obtain the equation[11]

$$w_{\xi\xi} + i\mu w_\xi - 2\frac{\bar{w}(w_\xi^2 + \alpha) - \alpha w^3 - \gamma w}{1 + |w|^2} - \gamma w = 0. \tag{3.14}$$

From this it is not difficult to obtain that

$$(w_\xi \bar{w} - \bar{w}_\xi w)_\xi + i\mu\,(|w|^2)_\xi - 2\frac{w_\xi^2\bar{w}^2 - \bar{w}_\xi^2 w^2}{1 + |w|^2} + 2\alpha(w^2 - \bar{w}^2) = 0. \tag{3.15}$$

Letting $w(\xi) = \rho\, e^{i\phi}$, where $\rho = \rho(\xi)$, $\phi = \phi(\xi)$, $\rho \in \mathbb{R}_+$, $\phi \in \mathbb{R}$, we then have:

$$2(\rho^2)_\xi\phi_\xi + 2\rho^2\phi_{\xi\xi} + \mu(\rho^2)_\xi - 8\frac{\rho^3\rho_\xi\,\phi_\xi}{1 + \rho^2} + 4\alpha\sin 2\phi = 0. \tag{3.16}$$

Let $\mu \neq 0$. Then, obviously, $\rho = const$, $\phi = \pi n/2$, $n = 0, \pm 1, \pm 2, \pm 3$, satisfy (3.16). For $\phi = \phi_0 = const$ we obtain, $\rho^2 = \rho_0^2 - (4\alpha/\mu)\sin(2\phi_0)\xi$, where $\rho_0 = const$; при $\rho = \tilde{\rho}_0 = const$ (3.16) is reduced to the equation of the pendulum: $\phi_{\xi\xi} + (2\alpha/\tilde{\rho}_0)\sin(2\phi) = 0$ (the existence of other solutions remains an open problem).

Let now $\mu = 0$, then from (3.16) it follows that

$$\frac{(\rho^2)_\xi}{\rho^2} - \frac{4\rho\rho_\xi}{1 + \rho^2} = C_1, \qquad \phi_{\xi\xi} + C_1\phi_\xi + 2\alpha\sin 2\phi = 0, \tag{3.17}$$

where C_1 is arbitrary constant. The first of these equations can easily be integrated:

$$\rho_{1,2}(\xi) = \frac{1}{2}(1 \pm \sqrt{1 - 4e^{-2(C_1\xi + C_2)}}\,)\, e^{C_1\xi + C_2}, \tag{3.18}$$

where C_2 is another arbitrary constant (we assume that $e^{-2(C_1\xi + C_2)} < 1/4$), and the second equation, which coincides with the one of the pendulum with the friction [12], admits, in particular, solutions of the form $\phi = \pi n/2$, $n = 0, \pm 1, \pm 2, \pm 3$.

Thus, solutions of the stationary Landau - Lifshits equation have fairly non-trivial structure in the generic (fully anisotropic) case. Their further study could bring a solution

[11] Stationary equations of another form for the Landau-Lifshits hierarchy were considered from the viewpoint of the Lie-algebraic approach in [24].

[12] In the partial case $C_1 = 0$ this equation, can obviously be integrated in terms of the elliptic functions.

to an important problem in the theory of dynamical systems - that of construction of the phase graph for the equations (3.14) and (3.6)[13]. The same applies to the deformed Heisenberg magnet from the previous section.

4. ISHIMORI MAGNET

a). *Physical and geometrical interpretations.*

The Ishimori magnet model in terms of the magnetization vector has the form:

$$\mathbf{S}_t = \mathbf{S} \wedge (\mathbf{S}_{xx} + \alpha^2 \mathbf{S}_{yy}) + u_y \mathbf{S}_x + u_x \mathbf{S}_y, \tag{4.1}$$

$$u_{xx} - \alpha^2 u_{yy} = -2\alpha^2 \mathbf{S}(\mathbf{S}_x \wedge \mathbf{S}_y), \tag{4.2}$$

where $\mathbf{S}(x, y, t) = (S_1, S_2, S_3)$ is a three dimensional vector, $|\mathbf{S}| = 1$, $u = u(x, y, t)$ is an auxiliary scalar real-valued field, and the parameter α^2 takes values ± 1. The system is called the Ishimori-I magnet (MI-I) in the case $\alpha^2 = 1$, the Ishimori-II magnet (MI-II) in the case $\alpha^2 = -1$. Mathematically, each of these cases corresponds to different types of the equations (4.1) and (4.2).

The topological charge of the model (4.1)-(4.2),

$$Q_T = \frac{1}{4\pi} \int_{\mathbb{R}^2} \int \mathbf{S}(\mathbf{S}_x \wedge \mathbf{S}_y)\, dx\, dy, \tag{4.3}$$

is invariant under the evolution of the system. Since the homotopy group of the unit 2-sphere $\pi_2(\tilde{S}^2)$ coincides with the group $\mathbb{Z}$ of integers, the number Q_T must be integer. According to (4.3), the scalar function $u = u(x, y, t)$ is related to the density of the topological charge production. The derivatives u_x, u_y in (4.1) play role of friction coefficients. Thus, (4.1) can be interpreted as an equation of forced (by the friction power) precession of the magnetization vector, and the system (4.1)-(4.2) is self-consistent.

From the physical viewpoint, it is easy to see that there is a non-local interaction in this system, on top of a local (exchange) one. The mechanism of the former is unclear. Nevertheless, the study of such systems is justified since stable localized two-dimensional magnetic structures are observed in experiments. An argument in favor of this assertion is the above-mentioned gauge equivalence of the MI-II model and the DS-II model, which describes quasi-monochromatic waves on the fluid surface [5], and also a link found in [26] between the MI-I model and the nonlinear Schrödinger equation with magnetic field.

Also helpful is another, hydrodynamical, interpretation of the model (4.1)-(4.2). Namely, let $u_y = -v_1$, $u_x = v_2$, hence $\mathbf{v}(x, y) = (v_1,\ v_2)$ is the velocity field of a fluid. Then the MI model can be rewritten as follows:

$$\mathbf{S}_t + v_1 \mathbf{S}_x - v_2 \mathbf{S}_y = \mathbf{S} \wedge (\mathbf{S}_{xx} + \alpha^2 \mathbf{S}_{yy}), \tag{4.4}$$

[13] Phase graphs of the equation (3.1) in the case of partial anisotropy have been studied in [25]. Phase graphs in the fully anisotropic case have apparently not been considered yet.

$$v_{2x} + \alpha^2 v_{1y} = -2\alpha^2 \mathbf{S}(\mathbf{S}_x \wedge \mathbf{S}_y).$$

If we define the stream function of the flow, $v_1 = -\chi_{1y}$, $v_2 = \chi_{1x}$, then the equation (4.2) with $\alpha^2 = -1$ (the MI-II model) implies the Poisson equation

$$\chi_{1xx} + \chi_{1yy} = 2\mathbf{S}(\mathbf{S}_x \wedge \mathbf{S}_y), \tag{4.5}$$

that is, the stationary (the time t is a parameter here) vorticity equation with a source in the right hand side of the magnitude proportional to the density of the topological charge production (details on the equation of planar hydrodynamical vortex can be found in [27]).

Let $\tilde{F}(x, y, t) = 2\mathbf{S}(\mathbf{S}_x \wedge \mathbf{S}_y)$. On taking one of the expressions of the form $\pm e^{\pm\chi_1}$, $e^{\chi_1} - e^{-2\chi_1}$, $\pm \sinh \chi_1$, $\pm \cosh \chi_1$, $\pm \sin \chi_1$, $\pm \cos \chi_1$, for $\tilde{F}(x, y, t)$, we obtain a closed completely integrable equation of elliptic type for the function χ_1. The solution of an appropriate boundary-value problem for this equation must satisfy the additional condition

$$\frac{1}{8\pi} \int_{\mathbb{R}^2}\!\!\int \triangle\chi_1(x, y) dx\, dy = N_0, \quad N_0 \in \mathbb{Z}, \tag{4.6a}$$

or $(r = \sqrt{x^2 + y^2})$

$$\lim_{r\to\infty} \frac{1}{8\pi} \oint (\chi_{1x} dy - \chi_{1y} dx) = N_0. \tag{4.6b}$$

b). *New canonical variables.*

Let us now consider another canonical variables. First, we pass from the variable $\mathbf{S}$ to new variables p и q $(p,\ q \in \mathbb{R})$ in (4.1)-(4.2), setting [28]:

$$S_3(x, y, t) = p(x, y, t), \quad S_+(x, y, t) = \sqrt{1 - p^2(x, y, t)}\, e^{iq(x,y,t)}. \tag{4.7}$$

Expressions for Poisson brackets of the quantities p и q follow directly from (2.2), on taking into account that the problem is two-dimensional,

$$\{p(\mathbf{r}), q(\mathbf{r}')\} = \delta(\mathbf{r} - \mathbf{r}'), \ \{p(\mathbf{r}), p(\mathbf{r}')\} = \{q(\mathbf{r}), q(\mathbf{r}')\} = 0, \quad \mathbf{r} = (x, y), \tag{4.8}$$

and then for the any two functionals F and G one can obtain:

$$\{F, G\} = \int_{\mathbb{R}^2}\!\!\int [\frac{\delta F}{\delta p(\mathbf{r})}\frac{\delta G}{\delta q(\mathbf{r})} - \frac{\delta F}{\delta q(\mathbf{r})}\frac{\delta G}{\delta p(\mathbf{r})}]\, dx\, dy. \tag{4.9}$$

In terms of this variables the MI model (4.1)-(4.2) can be rewritten as a Hamiltonian system,

$$\begin{aligned} q_t = \frac{\delta H}{\delta p} &= -\frac{p_{xx} + \alpha^2 p_{yy}}{1 - p^2} - \frac{p\,(p_x^2 + \alpha^2 p_y^2)}{(1 - p^2)^2} - p\,(q_x^2 + \alpha^2 q_y^2) + u_y q_x + u_x q_y, \\ p_t = -\frac{\delta H}{\delta q} &= (1 - p^2)(q_{xx} + \alpha^2 q_{yy}) - 2p\,(p_x q_x + \alpha^2 p_y q_y) + u_y p_x + u_x p_y, \\ u_{xx} - \alpha^2 u_{yy} &= -2\alpha^2 (p_y q_x - p_x q_y), \end{aligned} \tag{4.10}$$

and for the topological charge we will have:

$$Q_T = \frac{1}{4\pi}\int_{\mathbb{R}^2}\int (p_y q_x - p_x q_y)\, dx\, dy. \tag{4.11}$$

Here the Hamiltonian H has the form [14] :

$$\begin{gathered} H = H_1 + H_2, \quad H_1 = \frac{1}{2}\int_{\mathbb{R}^2}\int [\frac{p_x^2 + \alpha^2 p_y^2}{1-p^2} + (1-p^2)(q_x^2 + \alpha^2 q_y^2)]dxdy, \\ H_2 = \frac{1}{4}\int_{\mathbb{R}^2}\int [\alpha^2 A^2 + B^2]\, dx\, dy, \end{gathered} \tag{4.12}$$

where $A = u_x$, $B = -\alpha^2 u_y$, so that $A_x + B_y = 2\alpha^2(p_x q_y - p_y q_x)$; in this case it can be take place the conditions:

$$\begin{gathered} \frac{\delta A}{\delta p} = C\delta_y(y-y')\delta(x-x'), \quad \frac{\delta B}{\delta p} = D\delta(y-y')\delta_x(x-x'), \\ \frac{\delta A}{\delta q} = E\delta_y(y-y')\delta(x-x'), \quad \frac{\delta B}{\delta q} = F\delta(y-y')\delta_x(x-x'), \end{gathered} \tag{4.13}$$

where C, D, E, F are some functions. Letting $D = C$, $F = E$ and taking into account (4.10), we obtain the following relations on the functions $C = C(x, y, t)$ and $E = E(x, y, t)$ (the symbol $<, >$ refers to the scalar product in $\mathbb{R}^2$, and T stands for the transposition):

$$\begin{gathered} < \nabla u, \, (\frac{1}{2\alpha^2}q_y + C_y, \, \frac{1}{2\alpha^2}q_x - C_x)^T >= 0, \\ < \nabla u, \, (-\frac{1}{2\alpha^2}p_y + E_y, \, -\frac{1}{2\alpha^2}p_x - E_x)^T >= 0, \end{gathered} \tag{4.14}$$

from this we find:

$$\begin{gathered} C(x, y, t) = C_0(u(x, y)) + \frac{1}{2\alpha^2}\int_s (u_x q_y + u_y q_x) ds, \\ E(x, y, t) = E_0(u(x, y)) + \frac{1}{2\alpha^2}\int_s (u_x p_y + u_y p_x) ds, \end{gathered} \tag{4.15}$$

where C_0, E_0 are arbitrary functionals, and the integration goes along the characteristic s of the equations (4.14). Assuming that $E_0 = C_0$, we see that the functional C_0 must obey an additional condition:

[14]Paper [29] contains an expression for the Hamiltonian of the so-called modified MI different from (4.1) by the sign in the last but one term. Thus, the Hamiltonian for the model (4.1)-(4.2) seems to have been obtained here for the first time, both in the q, p and w, $\bar{w}$ variables, the latter being defined below. Also, in contrast with the modified model, it is easy to see that it is impossible to define the Clebsch variables in our case.

$$\frac{\delta C_0}{\delta u}\,(\frac{\delta u}{\delta q}-\frac{\delta u}{\delta p})=\frac{2}{\alpha^2}u_{xy}. \tag{4.16}$$

Since $u_{xy}\neq 0$ in the generic case, from this it follows that one more condition is necessary: $\delta C_0/\delta u \neq 0$ (if, of course, at this $\delta u/\delta q \neq \delta u/\delta p$).

Now let us pass to the variable w in (4.1)-(4.2), defined in (2.6) (assuming that $w = w(x,y,t)$) [15] :

$$\begin{gathered} iw_t = w_{xx}+\alpha^2 w_{yy} - 2\frac{\bar{w}(w_x^2+\alpha^2 w_y^2)}{1+|w|^2} + i(u_x w_y + u_y w_x), \\ u_{xx}-\alpha^2 u_{yy} = 4i\alpha^2\frac{w_x\bar{w}_y - \bar{w}_x w_y}{(1+|w|^2)^2}. \end{gathered} \tag{4.17}$$

Then for the topological charge we obtain

$$Q_T = -\frac{i}{2\pi}\int_{\mathbb{R}^2}\!\!\int \frac{w_x\bar{w}_y - \bar{w}_x w_y}{(1+|w|^2)^2}\,dx\,dy. \tag{4.18}$$

The non-vanishing of the Poisson bracket for the canonical variables $w(x,y), \bar{w}(x,y)$ comes along as in (2.8):

$$\{w(x,y),\bar{w}(x',y')\} = -\frac{i}{2}(1+|w|^2)^2\delta(\mathbf{r}-\mathbf{r}'), \quad \mathbf{r}=(x,y). \tag{4.19}$$

This allows to rewrite (4.1)-(4.2) in a transparently Hamiltonian form:

$$iw_t = -\frac{1}{2}(1+|w|^2)^2\,\frac{\delta H}{\delta \bar{w}}. \tag{4.20}$$

Here H is the Hamiltonian of the form

$$\begin{gathered} H = H_1 + H_2, \quad H_1 = 2\int_{\mathbb{R}^2}\!\!\int \frac{w_x\bar{w}_x + \alpha^2 w_y\bar{w}_y}{(1+|w|^2)^2}\,dx\,dy, \\ H_2 = \frac{1}{4}\int_{\mathbb{R}^2}\!\!\int [\alpha^2 u_x^2 + u_y^2]\,dx\,dy, \end{gathered} \tag{4.21}$$

and we assume in the course of the derivation of the equations for the model that the following conditions, analogous to (4.13), are satisfied:

$$\frac{\delta u_x}{\delta \bar{w}} = -\frac{4iw_x}{\alpha^2(1+|w|^2)^2}\,\delta(x-x')\,\delta(y-y'), \quad \frac{\delta u_y}{\delta \bar{w}} = -\frac{4iw_y}{(1+|w|^2)^2}\,\delta(x-x')\,\delta(y-y'). \tag{4.22}$$

Clearly, all three representations of the MI model, (4.1)-(4.2), (4.10) и (4.17) are equivalent.

[15] The reflection $(w,\,\bar{w}) \to (p,\,q)$ can by given by relations $q = -\arctan(i(w-\bar{w})/(w+\bar{w}))$, $p = (|w|^2-1)/(1+|w|^2)$.

Notice also that, one can define a "complex extension" of the system (4.17) analogous to the ones above. Letting formally $w_1 = \bar{w}$, one obtains,

$$
\begin{aligned}
iw_t &= w_{xx} + \alpha^2 w_{yy} - 2\frac{w_1(w_x^2 + \alpha^2 w_y^2)}{(1+ww_1)^2} + i(u_x w_y + u_y w_x), \\
iw_{1t} &= -(w_{1xx} + \alpha^2 w_{1yy}) + 2\frac{w(w_{1x}^2 + \alpha^2 w_{1y}^2)}{(1+ww_1)^2} + i(u_x w_{1y} + u_y w_{1x}), \\
u_{xx} - \alpha^2 u_{yy} &= 4i\alpha^2 \frac{w_x w_{1y} - w_{1x} w_y}{(1+ww_1)^2}.
\end{aligned}
\tag{4.23}
$$

This system can be interpreted as a model of two coupled Ishimori magnets. Nontrivial Poisson brackets follow from (4.19):

$$
\{w(x,y), w_1(x',y')\} = -\frac{i}{2}(1+ww_1)^2\delta(\mathbf{r}-\mathbf{r}'), \ \{\bar{w}(x,y), \bar{w}_1(x',y')\} = \frac{i}{2}(1+\bar{w}\bar{w}_1)^2\delta(\mathbf{r}-\mathbf{r}'),
\tag{4.24}
$$

and the "topological charge" of this model is [16]

$$
Q_T = -\frac{i}{2\pi}\int_{\mathbb{R}^2}\int \frac{w_x w_{1y} - w_{1x} w_y}{(1+ww_1)^2}\, dx\, dy.
\tag{4.25}
$$

The equations of motion (4.23) are Hamiltonian:

$$
iw_t = -\frac{1}{2}(1+ww_1)^2\frac{\delta H}{\delta w_1}, \qquad iw_{1t} = \frac{1}{2}(1+ww_1)^2\frac{\delta H}{\delta w},
\tag{4.26}
$$

where

$$
\begin{aligned}
H = H_1 + H_2, \quad H_1 &= 2\int_{\mathbb{R}^2}\int \frac{w_x w_{1x} + \alpha^2 w_y w_{1y}}{(1+ww_1)^2}\, dx\, dy, \\
H_2 &= \frac{1}{4}\int_{\mathbb{R}^2}\int [\alpha^2 u_x^2 + u_y^2]\, dx\, dy,
\end{aligned}
\tag{4.27}
$$

and we suppose that

$$
\begin{aligned}
\frac{\delta u_x}{\delta w_1} &= -\frac{4iw_y}{\alpha^2(1+ww_1)^2}, \qquad \frac{\delta u_y}{\delta w_1} = -\frac{4iw_x}{(1+ww_1)^2}, \\
\frac{\delta u_x}{\delta w} &= \frac{4iw_{1y}}{\alpha^2(1+ww_1)^2}, \qquad \frac{\delta u_y}{\delta w} = \frac{4iw_{1x}}{(1+ww_1)^2}.
\end{aligned}
\tag{4.27a}
$$

Returning to (4.17), we introduce the complex coordinates $z = x+iy$, $\bar{z} = x - iy$, so that $\partial_z = 1/2(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = 1/2(\partial_x + i\partial_y)$, $dx\, dy = (i/2)dz \wedge d\bar{z}$ and rewrite (4.18) and (4.21) in terms of these variables.

i). Let $\alpha^2 = 1$, that is, the MI-I model is considered. In this case we obtain:

[16] In general, $w_1 \neq \bar{w}$, thus Q_T can be non-integer and even complex. Such a situation, including an interpretation of the quantity Q_T, should be considered separately.

$$iw_t = 4w_{z\bar{z}} - 8\frac{w_z w_{\bar{z}}}{1+|w|^2}\bar{w} - 2(w_z u_z - w_{\bar{z}} u_{\bar{z}}),$$

$$u_{zz} + u_{\bar{z}\bar{z}} = 4\frac{w_z \bar{w}_{\bar{z}} - w_{\bar{z}} \bar{w}_z}{(1+|w|^2)^2}. \tag{4.28}$$

The topological charge is given by

$$Q_T = \frac{i}{2\pi}\int\int \frac{w_z \bar{w}_{\bar{z}} - w_{\bar{z}}\bar{w}_z}{(1+|w|^2)^2}\, dz \wedge d\bar{z}, \tag{4.29}$$

and the Hamiltonian is

$$H = 2i\int\int \frac{w_z \bar{w}_{\bar{z}} + w_{\bar{z}}\bar{w}_z}{(1+|w|^2)^2}\, dz \wedge d\bar{z} + \frac{i}{2}\int\int u_z u_{\bar{z}}\, dz \wedge d\bar{z}. \tag{4.30}$$

ii). Let $\alpha^2 = -1$. We then have the MI-II model,

$$iw_t = 2(w_{zz} + w_{\bar{z}\bar{z}}) - 4\frac{w_z^2 + w_{\bar{z}}^2}{1+|w|^2}\,\bar{w} - 2(w_z u_z - w_{\bar{z}} u_{\bar{z}}),$$

$$u_{z\bar{z}} = -2\frac{w_z \bar{w}_{\bar{z}} - w_{\bar{z}}\bar{w}_z}{(1+|w|^2)^2}. \tag{4.31}$$

The expression for the topological charge coincides with (4.29), and for the Hamiltonian we have:

$$H = 2i\int\int \frac{w_z \bar{w}_z + w_{\bar{z}}\bar{w}_{\bar{z}}}{(1+|w|^2)^2}\, dz \wedge d\bar{z} - \frac{i}{4}\int\int (u_z^2 + u_{\bar{z}}^2)\, dz \wedge d\bar{z}. \tag{4.32}$$

Notice also that the Hamiltonian of the MI-I magnet and its topological charge are related, as follows from (4.29) and (4.30), by the inequality of Bogomol'nyi, which is a lower estimate for the Hamiltonian taking into account all the dynamical configurations. Namely,

$$H \geq 4\pi Q_T. \tag{4.33}$$

Comparing (4.29) and (4.32) one can see that for the MI-II model such an estimate does not exist.

c). *Hamiltonians and topological charges for some of the simplest solutions.* [17]

Equations (4.1)-(4.2) can be interpreted as the compatibility conditions for the following overdetermined matrix systems on the function $\Psi = \Psi(x, y, t)$:

$$\Psi_y = \frac{1}{\alpha} S\Psi_x, \tag{4.34}$$

$$\Psi_t = -2iS\Psi_{xx} + Q\Psi_x, \tag{4.35}$$

[17] Similar calculations were given in [29] for the case of the modified MI and certain other systems.

where $Q = u_y I + \alpha^3 u_x S + i\alpha S_y S - iS_x$, $\Psi = \Psi(x, y, t) \in Mat(2, \mathbb{C})$, $S = \sum_{i=1}^{3} S_i \sigma_i$, σ_i are the standard Pauli matrices, I is the unit 2×2 matrix. By the definition, the S matrices have the properties: $S = S^*$, $S^2 = I$, $\det S = -1$, $\mathrm{Sp}\, S = 0$ (the asterisk stands for the Hermitian conjugation).

For a future reference, let us provide an expression for the S matrices in terms of the w variable:

$$S = \begin{pmatrix} \frac{|w|^1 - 1}{1+|w|^2} & \frac{2\bar{w}}{1+|w|^2} \\ \frac{2w}{1+|w|^2} & -\frac{|w|^2 - 1}{1+|w|^2} \end{pmatrix}, \tag{4.36}$$

and consider some of the simplest examples of calculations.

1. Let $\alpha^2 = -1$ in (4.17), that is, the MI-II model is considered. As was shown in [26], the conditions

$$w_{\bar{z}} = 0, \qquad w_z = 0, \tag{4.37}$$

are then compatible with (4.31). The first and second conditions here mean the presence of instanton and anti-instanton sectors, respectively, in the MI-II model. Consider the instanton sector assuming that $w(z) = ((z - z_0)/\lambda)^n$ [30][18], where $n \in \mathbb{Z}_+$, $\lambda \in \mathbb{C}$ (the z_0 and λ characterize, respectively, the position and size of the instanton). A calculation by the relation (4.39) gives:

$$Q_T = \frac{i}{2\pi} \int\int \frac{|w_z|^2}{(1 + |w|^2)^2}\, dz \wedge d\bar{z} = n, \tag{4.38}$$

and $H_1 = 0$ by (4.32). To find H_2, one has to know the function u. Using the second equation in (4.31) with $w_{\bar{z}} = 0$ and returning to the Cartesian coordinates, we obtain $(z_0 = x_0 + iy_0)$,

$$\triangle u = -8\lambda^{2n} \frac{[(x - x_0)^2 + (y - y_0)^2]^n}{[\lambda^{2n} + [(x - x_0)^2 + (y - y_0)^2]^n]^2}, \tag{4.39}$$

which implies that [19]

$$u(x, y) = -8\lambda^{2n} \int_{\mathbb{R}^2}\int G_0(x - x', y - y') \frac{[(x' - x_0)^2 + (y' - y_0)^2]^n}{[\lambda^{2n} + [(x' - x_0)^2 + (y' - y_0)^2]^n]^2}\, dx'\, dy', \tag{4.40}$$

where $G_0(x, y) = (1/2\pi)\ln(x^2 + y^2)$ is the Green function of the two-dimensional Laplace operator. Thus, the energy of the instanton solution on the formal level is given by [20]

$$H = \frac{1}{4} \int_{\mathbb{R}^2}\int (u_y^2 - u_x^2)\, dx\, dy. \tag{4.41}$$

The whole instanton sector is then split into disjoint classes each corresponding to the relevant value of the Q_T quantity.

[18] The choice of a more general solution, say, in the form of the Belavin-Polyakov instanton (linear-fractional function with complex poles) [2], unfortunately, significantly complicates the calculations.

[19] The following integral can be simplified by a change of variables and subsequent contour integration, but the remaining integral, apparently, cannot be calculated explicitly.

[20] Obviously, the Hamiltonian is positive in the domain where $|u_y| > |u_x|$.

2. Let us consider the MI-I model ($\alpha^2 = 1$) and show that instanton solutions exist in there as well[21],[22]. Indeed, for $w_{\bar{z}} = 0$ the system (4.28) is reduced to the following one,

$$w_t = -2u_z w_z, \quad u_{zz} + u_{\bar{z}\bar{z}} = 4\frac{w_z \bar{w}_{\bar{z}}}{(1+|w|^2)^2}. \tag{4.42}$$

Differentiating the first relation in $\bar{z}$, we obtain that $u_{z\bar{z}} = 0$, whence the compatibility is achieved if

$$u_{xx} = 4\frac{w_z \bar{w}_{\bar{z}}}{(1+|w|^2)^2}, \tag{4.43}$$

or

$$u(x,y,t) = 4\int_{-\infty}^{x} dx' \int_{-\infty}^{x'} dx'' \frac{|w_z|^2}{(1+|w|^2)^2} + f_0(y,t)x + c_1, \tag{4.44}$$

where c_1 is an arbitrary constant, and $f_0(.,.)$ is an arbitrary function. For the instanton solution $w(z)$ of the same form as in the previous case the number $Q_T = 0$ and the function

$$u(x,y,t) = \frac{4n^2}{\lambda^{2n}} \int_{-\infty}^{x} dx' \int_{-\infty}^{x'} dx'' \frac{[(x''-x_0)^2 + (y-y_0)^2]^{n-1}}{[\lambda^{2n} + (|x''-x_0|^2 + |y-y_0|^2)^n]^2} + f_0(y,t)x + c_1. \tag{4.45}$$

The expression for the Hamiltonian takes the form,

$$H = 4\pi n + \frac{1}{4}\int_{\mathbb{R}^2}\int (u_x^2 + u_y^2)\, dx\, dy. \tag{4.46}$$

3. Let us calculate the topological charge and the Hamiltonian for a solution of the form of a spiral structure. Namely, let $\mathbf{S} = (0,\, \sin\Phi_1,\, \cos\Phi_1)$, where $\Phi_1 = \delta_0 t + \alpha_0 x + \beta_0 y + \gamma_0$, α_0, β_0, γ_0, $\delta_0 \in \mathbb{R}$ are parameters, that is, the solution is a two-dimensional spiral structure [16]; then, according to (2.6) (see also (4.36)), $w(z,\bar{z}) = i\tan(\Phi_1/2)$, $\Phi_1 = \delta_0 t + \alpha z + \bar{\alpha}\bar{z} + \gamma_0$, $\alpha = \alpha_0/2 + \beta_0/(2i)$. It follows from (4.29) that $Q_T = 0$.

To determine the function $u = u(x,\, y,\, t)$, one has to substitute the function $w(z,\bar{z}))$ in the equations (4.28), (4.31), which gives two linear equations for u. Assuming their compatibility and integrating, we find ($\alpha^2 = -1$)[16]:

$$u(x,y) = g_0(y + \frac{\beta_0}{\alpha_0}x) + \int_s g_1(y(s') + \frac{\beta_0}{\alpha_0}x(s'), t)\, ds', \tag{4.47}$$

where g_0, g_1 are arbitrary functions such that g_0 is constant on the characteristic $y + (\beta_0/\alpha_0)x = const$, and s is the characteristic taken to be the integration path.

Similarly, for $\alpha^2 = 1$ we have:

[21] This is not surprising, albeit apparently went unnoticed in the literature, given that in the "static limit" the MI-I model turns into the elliptic version of the nonlinear $O(3)$ σ-model for which the instanton solutions were constructed initially. Notice also that the model was solved by the inverse scattering method in [31]-[32].

[22] From the viewpoint of the higher-dimensional inverse scattering method and the dressing procedures for solutions, the characteristic variables $\xi = (y-x)/2$ and $\eta = (x+y)/2$ [14], [15] are more natural than z and $\bar{z}$.

$$u(x,y) = g_2(y - \frac{\beta_0}{\alpha_0}x) + \int_{s_1} g_3(y(s') - \frac{\beta_0}{\alpha_0}x(s'), t)\, ds', \tag{4.48}$$

where g_2, g_3 are arbitrary functions, and g_2 is constant on the characteristic s_1 $y - (\beta_0/\alpha_0)x = const$.

The substitution $w = w(z,\ \bar{z})$ in (4.30) and (4.32) (in both MI-I and MI-II cases) leads to divergence of the Hamiltonian H_1, and, therefore, that of the Hamiltonian H as a whole, since the functional H_2 is finite.

4. As was first shown in [33] (see also [14], [16]), in the reflectionless section of the MI-II model the system (4.34)-(4.35) can be written in the form

$$\tilde{\Psi}_{\bar{z}} = 0, \quad \tilde{\Psi}_t + 2i\tilde{\Psi}_{zz} = 0. \tag{4.49}$$

In turn, the latter system has well-known polynomial solutions describing vortex states, $(\tilde{\Psi} = \{\tilde{\Psi}_{ij}\}, i,j = 1,2,\ \tilde{\Psi}_{22} = \bar{\tilde{\Psi}}_{11},\ \tilde{\Psi}_{12} = -\bar{\tilde{\Psi}}_{21})$[33]:

$$\begin{aligned} \tilde{\Psi}_{11}(z,t) &= \sum_{j=0}^{N_1} \sum_{m+2n=j} \frac{a_j}{m!n!}(-\frac{1}{2}z)^m(-\frac{1}{2}it)^n, \\ \tilde{\Psi}_{21}(z,t) &= \sum_{j=0}^{M_1} \sum_{m+2n=j} \frac{b_j}{m!n!}(-\frac{1}{2}z)^m(-\frac{1}{2}it)^n, \end{aligned} \tag{4.50}$$

where N_1 is an integer, $M_1 = N_1 - 1$, a_j, b_j are complex numbers, and the inner summations run over all $m,\ n \geq 0$ such that $m + 2n = j$. In particular, in this simplest case $N_1 = 1$ it follows that

$$\tilde{\Psi}_{11} = a_0 + a_1 z,\ \ \tilde{\Psi}_{21} = b_0. \tag{4.51}$$

Let us employ now a dressing (say, the Darboux dressing [16]) relation for the matrix S [23], $\tilde{S} = \tilde{\Psi}S^{(1)}\tilde{\Psi}^{-1}$, where $S^{(1)}$ is the initial solution of the system (4.1)-(4.2), assuming that $S^{(1)} = \sigma_3$. This leads to the so-called "one-lump" stationary solution, which we write here in terms of the stereographic projection,

$$w(z,\bar{z}) = \frac{\bar{b}_0}{\bar{a}_1(\bar{z} - \bar{z}_0) + \bar{d}_0}, \tag{4.52}$$

where $d_0 = a_0 - a_1 z_0$, z_0 is the coordinate of the vortex center on the complex plan.

It is known [33], that $Q_T = 1$ for such a solution. Let us calculate the function $u = u(x,y)$. In the case under consideration the second equation in (4.31) is reduced to the following,

$$u_{z\bar{z}} = \frac{2|a_1|^2|b_0|^2}{(|a_1(z - z_0) + d_0|^2 + |b_0|^2)^2}. \tag{4.53}$$

From this we find

[23] Its structure is identical, at least, for all the models of magnets treated here and all known methods of their solutions.

$$u(x,y) = 2|a_1|^2|b_0|^2 \int_{\mathbb{R}^2}\int G_0(x-x', y-y') \frac{1}{(|a_1(z'-z_0)+d_0|^2+|b_0|^2)^2}\, dx'\, dy'. \quad (4.54)$$

Taking into account that $H_1 = 0$, we now obtain from (4.32) the Hamiltonian of the one-lump solution in the form

$$H = \int_{\mathbb{R}^2}\int (u_y^2 - u_x^2)\, dx\, dy. \quad (4.55)$$

The left hand side is positive in the planar domain where $|u_y| > |u_x|$ in the same way as in (4.41).

5. CONCLUSION

The results for the Ishimori magnet show, in particular, that the Hamiltonians and topological charges cannot always be calculated analytically even for the simplest solutions, and numerics are required for specific Cauchy problems. In this respect, it is especially interesting, in our opinion, to check the hypothesis [16] of possibility of a phase transition in the model which involves a change of topology and symmetry properties of the system.

Concerning the "extended" systems (2.12) and (4.23), we would like to point out that if the initial systems are gauge equivalent to the nonlinear Schrödinger equation with an integral nonlinearity [8,9] (the Davy-Stewartson-II model [5] in the IM-II case), then it is interesting and important to find objects gauge equivalent to the extended systems.

Overall, the representations considered in this paper can, hopefully, be useful in studying other (1+1)— (and more realistic (2+1)— and (3+1)—) dimensional models of magnets, σ-models and chiral fields, including nonintegrable cases.

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APPENDIX

We provide here an expression for a Lax pair for the system (2.12). First, define the function $D = D(x,t) \in Mat(2,\mathbb{C})$, $x > 0$, of the form

$$D(x,t) = \begin{pmatrix} \frac{sr-1}{1+sr} & \frac{2s}{1+sr} \\ \frac{2r}{1+sr} & -\frac{sr-1}{1+sr} \end{pmatrix}. \quad (A.1)$$

The matrix D has the following properties: $D^2 = I$, Sp $D = 0$, det $D = -1$. However, unlike the matrix S, it is not Hermitian. A straightforward calculation shows that (2.12) is the compatibility condition for the following overdetermined linear system of equations:

$$\Psi_x = -\frac{i}{2}\lambda D\Psi, \quad \Psi_t = (\frac{\lambda}{2}D_xD + \frac{i}{2}\lambda^2 D)\Psi, \tag{A.2}$$

where $\Psi = \Psi(x,t,\lambda) \in Mat(2,\mathbb{C})$, $\lambda = \lambda(x,t,u)$, $u \in \mathbb{C}$. This means that the parameter u plays the role of a "hidden" spectral parameter, so that the conditions

$$\lambda_x = \frac{\lambda}{x}, \qquad \lambda_t = -\frac{2\lambda^2}{x}, \tag{A.3}$$

or

$$\lambda = \frac{x}{2(t+u)}, \tag{A.4}$$

are fulfilled, and the matrices D satisfy the equation

$$iD_t = \frac{1}{2}[D, D_{xx}] - \frac{1}{x}D_xD. \tag{A.5}$$

Thus, we have a non-isospectral deformation of the associated linear system (the case of a single deformed Heisenberg magnet was considered in [8]). Notice also that so far the inverse scattering transform method has not been applied to such systems at the full scale.

REFERENCES

1. *L.D.Landau,* - Collected Papers, (1969), 128-143, Moscow, Nauka.
2. *A.A.Belavin and A.M.Polyakov,* - JETP Letters, 22 (1975), 503-506.
3. *A.M.Perelomov,* - Uspekhi Fizich. Nauk, 134 (1990), 577-609.
4. *Ju.M.Izumov and Ju.N.Skriabin,* - Statistical mechanics of the magnet-nonordered medium, (1987), Moscow, Nauka.
5. *V.D.Lipovskii and A.V.Shirokov,* - Funct. An. and Appl, 23 (1989), 65.
6. *O.I.Mokhov,* - Symplectic and Poisson geometry on the spaces of the loops of the smooth manifolds and integrable equations (2004), - Institute of the Computers' s researches, Moscow-Igevsk.
7. *A.V.Mikhailov and A.I.Jaremchuk,* - JETP Letters, 36 (1982), 78.
8. *S.P.Burtsev, V.E.Zakharov and A.V.Mikhailov,* - TMP, 70 (1987), 323.
9. *K.Porsezian and M.Lakshmanan,* - J.Math.Phys., 32, (1991), 2923.
10. *E.Sh.Gutshabash,* - JETP Letters, 73 (2001), 317-319.
11. *E.K.Sklyanin,* - Preprint No E-3-79 LOMI (1979), Leningrad.

12. *A.I.Bobenko,* - Zapiski Nauchnych Seminarov POMI, 123, (1983), 58.

13. *Niam-Ning Huang, Hao Cai and ot.,* - arxiv:nlin.SI/0509042.

14. *B.G.Konopelchenko,* - Solitons in Multidimensions, (1993), World Scientific.

15. *E.Sh.Gutshabash,* - Zapiski Nauchnych Seminarov POMI, 17 (2002), 155-168.

16. *E.Sh.Gutshabash,* - JETP Letters, 78 (2003), 1257.

17. *Qing Ding,* - arxiv.math.DG/0504288.

18. *A.N.Leznov and A.V.Razumov,* - J.Math.Phys., 35 (1994), 1738.

19. *E.V.Ferapontov,* - TMP, 91 (1992), 452.

20. *V.G.Mahan'kov, Ju.P.Rybakov and V.I.Sanuk,* - Uspekhi Fizich. Nauk, 162 (1994), 121-148.

21. *L.A.Takhtadjan and L.D.Faddeev,* - Hamiltonian approach in theory of Solitons (1986), Moscow, Nauka.

22. *E.Barouch, A.Fokas and V.Papageorgiou,* - J.Math.Phys., 29 (1988), 2628.

23. *E.M.Lifshits and L.P.Pitaevskii,* - Statistical Physics, part 2 (1978), Moscow, Nauka.

24. *V.G.Bariakhtar, E.D.Belokolos and P.I.Golod,* - In: Modern Problems in Magnetism, (1986), 30, Kiev, Naukova dumka.

25. *V.M.Eleonskii, N.N.Kirova and N.E.Kulagin,* - JETP, 79 (1980), 321.

26. *V.G.Mikhalev,* - Zapiski Nauchnych Seminarov POMI, 189 (1991), 75.

27. *E.Sh.Gutshabash,* - Zapiski Nauchnych Seminarov POMI, 19, (2006), 119-133.

28. *H.C. Fogedby,* - J.Phys.A: Math.Gen., 13 (1980), 1467-1499.

29. *L.Martina, G.Profilo, G.Soliani and L.Solombrino,* - Phys. Rev.B, 49 (1994), 12915.

30. *R.Rajaraman,* - Solitons and Instantons (1982), North-Holland Publishing Company, Amsterdam-New York-Oxford.

31. *E.Sh.Gutshabash and V.D.Lipovskii,* - TMP, 90 (1992), 175.

32. *G.G.Varzugin, E.Sh.Gutshabash and V.D.Lipovskii,* - TMP, 104 (1995), 513.

33. *Y.Ishimori,* - Progr.Teor.Phys, 72 (1984), 33.